DECOMPOSITIONS OF E^3 WITH A COMPACT 0-DIMENSIONAL SET OF NONDEGENERATE ELEMENTS(1)

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1. **Introduction.** The purpose of this paper is to study monotone upper semicontinuous decompositions G of E^3 such that the image, under the projection map, of the union of all the nondegenerate elements of G is contained in a compact 0-dimensional set. Such decompositions are of interest since a number of examples that have been studied [4], [6], [7], [9], [15] satisfy the conditions imposed above on G.

Suppose then that G is a monotone upper semicontinuous decomposition of E^3. Let E^3/G denote the associated decomposition space and let P denote the projection map from E^3 onto E^3/G. Let H_G denote the union of all the nondegenerate elements of G. Suppose that $P[H_G]$ is contained in a compact 0-dimensional set.

In §§3 and 4, point-like decompositions are considered. In §4, we prove that if G is point-like and E^3/G is a 3-manifold, then E^3/G is homeomorphic to E^3. This settles a special case of a question raised by Bing in [7]. Other special cases of this question have been settled in [1], [2], [14], and [19]. Also in §4, we give a condition which is both necessary and sufficient in order that E^3/G be homeomorphic to E^3 in case G is a point-like decomposition satisfying the conditions above. This condition is in terms of the existence of homeomorphisms from E^3 onto E^3 that shrink the nondegenerate elements of G to small size.

In §§ 5 and 6, we study the following question: Is it true that if G satisfies the conditions imposed above and E^3/G is homeomorphic to E^3, then each element of G is point-like? It is known [12] that if the set of nondegenerate elements is countable, the question has an affirmative answer. Furthermore, there is an example due to Bing [10, p. 7] of a monotone decomposition B of E^3 into non-point-like sets such that E^3/B is homeomorphic to E^3. In this case, $P[H_B]$ is an arc. Consequently, the condition imposed above on $P[H_G]$ cannot be omitted completely if an affirmative answer to the question is to be obtained. Although we do not settle this question, we give some partial affirmative solutions. We

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give a solution in case each nondegenerate element of G is an arc, or, in fact, any tree-like continuum. Additional partial solutions of this question are given in [3] and [13].

In §7, we give a number of special results concerning point-like decompositions G of E^3 such that E^3/G is homeomorphic to E^3.

2. **Notation and terminology.** If X is a topological space and G is an upper semi-continuous decomposition of X, then X/G denotes the associated decomposition space, P denotes the projection from X onto X/G, and H_G denotes the union of all the nondegenerate elements of G.

The statement that the upper semicontinuous decomposition G of E^3 is *monotone* means that each set of G is a compact continuum. A compact continuum K in E^3 is *point-like* if and only if E^3-K is homeomorphic to $E^3-\{0\}$. By a *point-like decomposition* of E^3 is meant a monotone decomposition of E^3 into point-like sets.

If n is a positive integer, the statement that M is an *n-manifold* means that M is a separable metric space, each point of which has a neighborhood U in M such that U is an open *n-cell.* The statement that M is an *n-manifold-with-boundary* means that M is a separable metric space such that each point of M has a neighborhood U in M such that U is an n-cell. *If* M is an n-manifold-with-boundary, then the *boundary* of M, denoted by Bd M, is the set of all points p of M wich do not have open n-cell neighborhoods in M, and the *interior* of M, Int M, is M − Bd M.

3. **Preliminary results.** The main result of this section, Theorem 1, will be used in §4 to construct homeomorphisms from E^3 onto E^3 having certain properties. Lemmas 1 and 2 are preliminary results for Theorem 1.

Suppose that C is a polyhedral 3-cell in E^3. If x and y are distinct points of Bd C, the statement that α is an *unknotted chord* of C from x to y means that α is a polygonal arc with endpoints x and y such that (1) (Int α) $\subset$ Int C and (2) if β is any polygonal arc on Bd C from x to y, then $\alpha \cup \beta$ is the boundary of a polyhedral disc D such that (Int D) $\subset$ Int C. It may be shown [20] that if α and α' are two unknotted chords of the polyhedral 3-cell C such that α and α' have the same endpoints, there is a homeomorphism h from C onto C such that if $p \in$ Bd C, $h(p) = p$ and $h[\alpha] = \alpha'$.

LEMMA 1. *If C is a polyhedral* 3-*cell in E^3, x and y are distinct points of* Bd C, *and B is a compact* 0-*dimensional subset of C containing neither x nor y, then there is an unknotted chord α of C such that α has endpoints x and y and is disjoint from B.*

Proof. We may assume that C is the cube

$$\{(x, y, z)\colon\ (x, y, z) \in E^3,\ \ |x| \leqq 1, |y| \leqq 1, \textit{and}\ |z| \leqq 1\},$$

that B intersects neither the top nor bottom face of C, and that x and y belong to the top and bottom faces, respectively, of C. Let D be the common part of C and the XZ-plane. D is a disc in C such that Bd $D \subset$ Bd C, and x and y lie on Bd D.

Since B is a compact 0-dimensional set, $B \cap D$ does not separate x and y in D. Hence there is polygonal arc α on D such that α has endpoints x and y, α is disjoint from B, and (Int α) $\subset$ Int D. It is easy to see that α is an unknotted chord of C.

The proof of Lemma 1 given above was shown to the author by R. H. Bing and is simpler than the author's original argument.

The following lemma is proved in [19].

LEMMA 2. *If G is a point-like decomposition of E^3 such that E^3/G is homeomorphic to E^3 and U is a simply connected open set in E^3/G, then $P^{-1}[U]$ is simply connected.*

We may now establish the main result of this section.

THEOREM 1. *Suppose that G is a point-like decomposition E^3 such that E^3/G is homeomorphic to E^3 and* Cl $P[H_G]$ *is a compact 0-dimensional set. Suppose that M is a compact polyhedral 3-manifold-with-boundary in E^3 such that* Bd M *is connected and is disjoint from* Cl H_G. *Then there is a homeomorphism h from M onto $P[M]$ such that* $h \mid \text{Bd } M = P \mid \text{Bd } M$.

Proof. Let K denote $P[\text{Bd } M]$; K is homeomorphic to Bd M. Since E^3/G is homeomorphic to E^3, K has exactly one bounded complementary domain U in E^3/G. Let N denote $K \cup U$. It may be shown that $P[M] = N$.

Since Bd M has a cartesian product neighborhood in E^3, K has a cartesian product neighborhood in E^3/G. Hence N is a compact 3-manifold-with-boundary, Bd $N = P[\text{Bd } M]$, and Int $N = P[\text{Int } M]$.

Since Theorem 1 is easily proved if Bd M is a 2-sphere, we assume that Bd M is not a 2-sphere. Let T_0 be a triangulation of N[11], [16] such that if σ is any 3-simplex of T_0 and S_2 is the carrier of the 2-skeleton on T_0, then $S_2 - \text{Bd } \sigma$ is connected. Let $v_1, v_2, \cdots$, and v_k be the vertices of T_0 lying in Int N. Let $B_1, B_2, \cdots$, and B_k be mutually disjoint open balls such that if $i = 1, 2, \cdots$, or k, $v_i \in B_i$ and $B_i \subset \text{Int } N$. If $i = 1, 2, \cdots$, or k, then, since Cl $P[H_G]$ is 0-dimensional, $B_i - \text{Cl } P[H_G]$ exists, and let v_i^* be a point of $B_i - \text{Cl } P[H_G]$. There is a homeomorphism f_0 from N onto N such that if $x \in N$ and $x \notin \bigcup_{i=1}^k B_i$, then $f_0(x) = x$, and if $i = 1, 2, \cdots$, or k, $f_0[B_i] = B_i$ and $f_0(v_i) = v_i^*$. Let T_1 be $\{f_0[\sigma] : \sigma \in T_0\}$. Then T_1 is a triangulation of N, no vertex of T_1 belongs to Cl $P[H_G]$, and if σ is any 3-simplex of T_1 and S_2' is the carrier of the 2-skeleton of T_1, then $S_2' - \text{Bd}\sigma$ is connected.

If no 1-simplex of T_1 intersects Cl $P[H_G]$, let T be T_1. Otherwise, let $s_1, s_2, \cdots$, and s_m be the 1-simplexes of T_1 which intersect Cl $P[H_G]$. There exist mutually disjoint polyhedral (relative to T_1) 3-cells $R_1, R_2, \cdots$, and R_m such that if $i = 1, 2, \cdots$, or m, then (1) $(s_i \cap \mathrm{Cl}\, P[H_G]) \subset \mathrm{Int}\, R_i$, (2) $R_i \subset \mathrm{Int}\, N$, and (3) $s_i \cap R_i$ is an unknotted chord of R_i. Such 3-cells may be constructed in the following way: If $i = 1, 2, \cdots$, or m, let s_i' be a subarc of s_i, lying in $\mathrm{Int}\, s_i$, and such that $(s_i' \cap \mathrm{Cl}\, P[H_G]) \subset \mathrm{Int}\, s_i'$. Since s_i' is polygonal (relative to T_1), it may be thickened slightly to give R_i.

If $i = 1, 2, \cdots$, or m, let x_i and y_i be the two points of $s_i \cap \mathrm{Bd}\, R_i$. By Lemma 1, there is an unknotted chord t_i of R_i from x_i to y_i missing $\mathrm{Cl}\, P[H_G]$. There is, then, a homeomorphism k_i from R_i onto R_i such that if $p \in \mathrm{Bd}\, R_i$, $k_i(p) = p$ and $k_i[s_i \cap R_i] = t_i$. It is clear that there is a homeomorphism k from N onto N such that (1) if $x \notin \bigcup_{i=1}^{m} R_i$, $k(x) = x$ and (2) if $i = 1, 2, \cdots$, or m, and $x \in R_i$, $k(x) = k_i(x)$. Let T be $\{k[\sigma] : \sigma \in T_1\}$. Then T is a triangulation of N such that the carrier Σ_1 of the 1-skeletion of T is disjoint from $\mathrm{Cl}\, P[H_G]$. Let Σ_2 denote the carrier of the 2-skeleton of T.

Let $\Delta_1, \Delta_2, \cdots$, and Δ_r be the 2-simplexes of T whose interiors lie in $\mathrm{Int}\, N$. If $i = 1, 2, \cdots$, or r, there is an annulus A_i on Δ_i such that (1) A_i is disjoint from $\mathrm{Cl}\, P[H_G]$ and (2) $\mathrm{Bd}\, \Delta_i$ is one boundary component of A_i. If $i = 1, 2, \cdots$, or r, let J_i be a centerline of A_i and let D_i be the subdisc of Δ_i having J_i as its boundary; $D_i \cap A_i$ is an annulus B_i. Let V_1, V_2, $\cdots$, and V_r be mutually disjoint simply connected open sets such that if $i = 1, 2, \cdots$, or r, then (1) $\mathrm{Cl}\, V_i \subset \mathrm{Int}\, N$, (2) $V_i \cap \Sigma_2 = \mathrm{Int}\, D_i$, and (3) $\mathrm{Cl}\, V_i \cap \Sigma_2 = D_i$. Such open sets may be constructed by slight thickenings of $\mathrm{Int}\, D_1$, $\mathrm{Int}\, D_2, \cdots$, and $\mathrm{Int}\, D_r$.

If $i = 1, 2, \cdots$, or r, let U_i be $P^{-1}[V_i]$. By Lemma 2, each of $U_1, U_2, \cdots$, and U_r is simply connected. It is easily seen, with the aid of Dehn's lemma [18] that if $i = 1, 2, \cdots$, or r, there is a polyhedral disc E_i such that (1) $\mathrm{Int}\, E_i \subset U_i$ and (2) $\mathrm{Bd}\, E_i = P^{-1}[J_i]$. Note that $E_i \subset M$. If $i = 1, 2, \cdots$, or r, let Y_1 be $E_i \cup P^{-1}[A_i - B_i]$; Y_i is a disc with boundary $P^{-1}[\Delta_i]$, and $Y_i \subset M$.

If i and j are distinct positive integers, neither greater than r, then $\mathrm{Int}\, Y_i$ and $\mathrm{Int}\, Y_j$ are disjoint, for (1) $\mathrm{Int}\, A_i$ and $\mathrm{Int}\, A_j$ are disjoint and (2) U_i and U_j are disjoint.

If $i = 1, 2, \cdots$, or r, let L_i denote the closure of $(A_i - B_i)$. It is clear that $P^{-1} \mid [(\mathrm{Bd}\, N) \cup (\bigcup_{i=1}^{r} L_i)]$ is a homeomorphism h_0 from $(\mathrm{Bd}\, N) \cup (\bigcup_{i=1}^{r} L_i)$ onto $(\mathrm{Bd}\, M) \cup (\bigcup_{i=1}^{r} P^{-1}[L_i])$ such that $h_0 \mid \mathrm{Bd}\, N = P^{-1} \mid \mathrm{Bd}\, N$. It is easily seen that there is a homeomorphism g from Σ_2 onto $(\mathrm{Bd}\, M) \cup (\bigcup_{i=1}^{r} Y_i)$ such that (1) $g \mid \mathrm{Bd}\, N = P^{-1} \mid \mathrm{Bd}\, N$ and (2) if $i = 1, 2, \cdots$, or r, $g[\Delta_i] = Y_i$. It is clear that $g[\Sigma_2] \subset M$.

Suppose that σ is a 3-simplex of T and let S be $\mathrm{Bd}\, \sigma$. We shall show now that $g[\Sigma_2]$ is disjoint from $\mathrm{Int}\, g[S]$, the interior, in E^3, of the 2-sphere $g[S]$.

Suppose that there is a point p of Σ_2 such that $g(p) \in \mathrm{Int}\, g[S]$. Clearly $p \in \Sigma_2 - S$. Now T has the property that if σ is any 3-simplex of T, then $\Sigma_2 - \mathrm{Bd}\, \sigma$ is connected. Therefore $\Sigma_2 - S$ is connected. Consequently $g[\Sigma_2] \subset (g[S] \cup \mathrm{Int}\, g[S])$, and thus $\mathrm{Bd}\, M \subset (g[S] \cup \mathrm{Int}\, g[S])$. Since $\mathrm{Bd}\, M$ is not a 2-sphere, there is a point of $g[S]$ not in M. This is a contradiction, for $g[\Sigma_2] \subset M$. Hence, no point of $g[\Sigma_2]$ lies in $\operatorname{Int} g[S]$.

Let $K_1, K_2, \cdots$, and K_n be the distinct 2-spheres in N which are boundaries of 3-simplexes of T. Let W be $\{q[K_i] \cup \operatorname{Int} g[K_i] : i = 1, 2, \cdots, \textit{or } n\}$. We shall show now that W is a triangulation of M.

First, W is a 3-complex. For suppose that i and j are distinct positive integers, neither greater than n. Then $\operatorname{Int} g[K_i]$ and $\operatorname{Int} g[K_j]$ are disjoint. For, it was shown that $g[K_i]$ and $\operatorname{Int} g[K_j]$ and $\operatorname{Int} g[K_j]$ are disjoint, and so are $g[K_j]$ and $\operatorname{Int} g[K_i]$. If $\operatorname{Int} g[K_j]$ and $\operatorname{Int} g[K_j]$ intersect, then one contains the other, say $\operatorname{Int} g[K_i] \subset \operatorname{Int} g[K_j]$. Since $g[K_i] \neq g[K_j]$, then $g[K_i]$ intersects $\operatorname{Int} g[K_j]$. This is a contradiction, and hence $\operatorname{Int} g[K_i]$ and $\operatorname{Int} g[K_j]$ are disjoint. Since $g[K_i] \cap g[K_j] = g[K_i \cap K_j]$, it is clear that W is a 3-complex.

Second, $\bigcup\{w : w \in W\} = M$. Since $g[\Sigma_2] \subset M$, it is clear that

$$\bigcup\{w : w \in W\} \subset M.$$

Hence we need only to show that $M \subset \bigcup\{w : w \in W\}$.

Suppose that $M = \bigcup\{w : w \in W)$ exists. Since $\bigcup\{w : w \in W\}$ is closed, there is a point p of $\operatorname{Int} M$ such that $p \notin \bigcup\{w : w \in W\}$. Let q be a point such that for some 3-simplex w of $W, q \in \operatorname{Int} w$. Then $q \in \operatorname{Int} M$ and there is an arc pq in $\operatorname{Int} M$; we may assume that pq is disjoint from the carrier of the 1-skeleton of W. Let b be the first point of $\bigcup\{w : w \in W\}$ on pq in the order from p to q; $pb - \{b\}$ is disjoint from $\bigcup\{w : w \in W\}$. There is a 2-simplex τ of W such that $b \in \operatorname{Int} \tau$, and there is a 3-simplex σ of W such that $\tau \subset \operatorname{Bd} \sigma$. Now $\tau \not\subset \operatorname{Bd} M$. For if $\tau \subset \operatorname{Bd} M$, then clearly the arc pb intersects $E^3 - M$, which is impossible. Thus $\tau \not\subset \operatorname{Bd} M$. It follows that $g^{-1}[\tau] \not\subset \operatorname{Bd} N$.

There is exactly one 3-simplex t_1 of T such that $\operatorname{Bd} \sigma = g[\operatorname{Bd} t_1]$. There is a 3-simplex t_2 of T such that $\operatorname{Int} t_1$ and $\operatorname{Int} t_2$ are disjoint and $\operatorname{Bd} t_1 \cap \operatorname{Bd} t_2 = g^{-1}[\tau]$. Let σ' be $g[\operatorname{Bd} t_2] \cup \operatorname{Int} g[\operatorname{Bd} t_2]$. Then $\sigma' \in W$ and $\sigma \cap \sigma' = \tau$. Since $\operatorname{Int} \sigma$ and $\operatorname{Int} \sigma'$ are disjoint and $\operatorname{Bd} \sigma \cap \operatorname{Bd} \sigma' = \tau$, it follows that $(pb - \{b\}) \subset \operatorname{Int} \sigma'$. This is a contradiction, and it follows that $M = \bigcup\{w : w \in W\}$.

Consequently, W is a triangulation of M. It is clear that there is an extension f of g such that (1) f is a homeomorphism from N onto M and (2) $f \mid \operatorname{Bd} N = P^{-1} \mid \operatorname{Bd} N$. It follows that there is a homeomorphism h from M onto N such that $h \mid \operatorname{Bd} M = P \mid \operatorname{Bd} M$. This concludes the proof of Theorem 1.

4. Results on point-like decompositions.

In this section, we establish the main results of the paper relative to point-like decompositions G of E^3 such that E^3/G either is homeomorphic to E^3 or is a 3-manifold.

THEOREM 2. *Suppose that G is a point-like decomposition of E^3 such that (1) E^3/G is homeomorphic to E^3 and (2) $\operatorname{Cl} P[H_G]$ is a compact 0-dimensional set. Suppose that U is an open set in E^3 containing $\operatorname{Cl} H_G$ and ε is a positive* *number. Then there is a homeomorphism h from* E^3 *onto* E^3 *such that* (1) *if* $x \notin U$, $h(x) = x$ *and* (2) *if g is any nondegenerate element of* G, $(\operatorname{diam} h[g]) < \varepsilon$.

Proof. Since $\operatorname{Cl} P[H_G]$ is a compact 0-dimensional subset of E^3/G and E^3/G is homeomorphic to E^3, there exists a sequence $N_1, N_2, N_3, \cdots$ such that (1) for each positive integer i, N_i is a compact polyhedral 3-manifold-with-boundary such that (a) each component of N_i has connected boundary and diameter less than $1/i$, and (b) $N_{i+1} \subset \operatorname{Int} N_i$, and (2) $\operatorname{Cl} P[H_G] = \bigcup_{i=1}^{\infty} N_i$. For each positive integer i, let M_i denote $P^{-1}[N_i]$; M_i is a compact 3-manifold-with-boundary. Since, for each positive integer i, N_i is polyhedral, then M_i is tame in E^3.

Since $\operatorname{Cl} H_G \subset U$, $P[U]$ is open in E^3/G and there is, accordingly, a positive integer k such that $N_k \subset P[U]$. It follows that $M_k \subset U$. Let $M_{k1}, M_{k2}, \cdots$, and M_{km} be the components of M_k. If $j = 1, 2, \cdots$, or m, there is, by Theorem 1, a homeomorphism f_j from M_{kj} onto $P[M_{kj}]$ such that $f_j | \operatorname{Bd} M_{kj} = P | \operatorname{Bd} M_{kj}$. Let f be the homeomorphism from M_k onto $P[M_k]$ such that if $j = 1, 2, \cdots$, or m, $f | M_{kj} = f_j$. Now f^{-1} is uniformly continuous and there exists a positive integer r greater than k such that if L is any component of $P[M_r]$, then $(\operatorname{diam} f^{-1}[L]) < \varepsilon$. Let $M_{r1}, M_{r2}, \cdots$, and M_{rn} be the components of M_r. With the aid of Theorem 1, it follows that if $i = 1, 2, \cdots$, or n, there is a homeomorphism k_i from M_{ri} onto $f^{-1}[P[M_{ri}]]$ such that $k_i | \operatorname{Bd} M_{ri} = f^{-1}P | \operatorname{Bd} M_{ri}$. Let k be the homeomorphism from M_r onto $f^{-1}[P[M_r]]$ such that if $i = 1, 2, \cdots$, or $n, k | M_{ri} = k_i$.

Define a homeomorphism h as follows: (1) If $x \notin M_k$, then $h(x) = x$. (2) If $x \in (M_k - M_r)$, then $h(x) = f^{-1}(P(x))$. (3) If $x \in M_r$, then $h(x) = k(x)$. Then h is a homeomorphism from E^3 onto E^3. If $x \notin U$, then since $x \notin M_k$, $h(x) = x$. If g is any nondegenarate element of G, then for some positive! integer i not greater than n, $g \subset M_{ri}$. Hence $h[g] \subset h[M_{ri}]$; but $h[M_{ri}] = f^{-1}[P[M_{ri}]]$, and $(\operatorname{diam} f^{-1}[PM_{ri}]]) < \varepsilon$. Therefore $(\operatorname{diam} h[g]) < \varepsilon$ and the proof of Theorem 2 is complete.

The following theorem may be compared with Theorem 2 of [1].

THEOREM 3. *Suppose that G is a point-like decomposition of* E^3 *such that* $P[H_G]$ *is a compact 0-dimensional set. Then* E^3/G *is homeomorphic to* E^3 *if and only if for each open set U in* E^3 *containing* H_G *and each positive number* ε, *there is a homeomorphism h from* E^3 *onto* E^3 *such that* (1) *if* $x \notin U$, $h(x) = x$ *and* (2) *if g is any nondegenerate element of* G, $(\operatorname{diam} h[g]) < \varepsilon$.

Proof. If, for each open set U in E^3 containing H_G and each positive number ε, there exists a homeomorphism h from E^3 onto E^3 having the properties stated above, then it follows from the proof of Theorem 1 of [4] that E^3/G is homeomorphic to E^3. Hence the condition stated is sufficient. By Theorem 2, it is necessary.

The following theorem may be compared with Theorem 5 of [1].

THEOREM 4. *Suppose that G is a point-like decomposition of E^3 such that $P[H_G]$ is a compact 0-dimensional set. If E^3/G is a 3-manifold, then E^3/G is homeomorphic to E^3.*

Proof. If E^3/G is a 3-manifold, then a local version of Theorem 1 holds. Hence the proof of Theorem 2 is valid and thus the conclusion of that theorem holds. Then by Theorem 3, E^3/G is homeomorphic to E^3.

COROLLARY 1. *If G is a point-like decomposition of S^3 such that $P[H_G]$ is a compact 0-dimensional set and S^3/G is a 3-manifold, then S^3/G is a 3-sphere.*

5. **Monotone decompositions of** E^3. In this section and the next, we study monotone decompositions G of E^3 such that (1) $P[H_G]$ is contained in a compact 0-dimensional set and (2) E^3/G is homeomorphic to E^3. We want to determine conditions under which it can be concluded that each element of G is point-like.

A compact continuum K in E^3 is *cellular* if and only if there exists a sequence $C_1, C_2, C_3, \cdots$ of 3-cells in E^3 such that (1) if n is any positive integer, $C_{n+1} \subset \text{Int } C_n$, and (2) $K = \bigcap_{i=1}^{\infty} C_i$. It is well known that for compact continua in E^3, "point-like" and "cellular" are equivalent; see [21].

THEOREM 5. *Suppose that G is a monotone decomposition of E^3 such that $P[H_G]$ is contained in a compact 0-dimensional set and E^3/G is homeomorphic to E^3. Suppose that if U is any open set in E^3 containing* Cl H_G *and ε is any positive number, there exists a homeomorphism f from E^3 onto E^3 such that* (1) *if $x \notin U$, $f(x) = x$, and* (2) *if g is any nondegenerate element of G, then* $(\text{diam} f[g]) < \varepsilon$. *Then each element of G is point-like.*

Proof. Since $\text{Cl}\, P[H_G]$ is compact and 0-dimensional, there exists a sequence $N_1, N_2, N_3, \cdots$ of compact 3-manifolds-with-boundary in E^3/G such that (1) if i is any positive integer, $N_{i+1} \subset \text{Int } N_i$, and each component of N_i has diameter less than $1/i$, and (2) $\bigcap_{i=1}^{\infty} = P[H_G]$. For each positive integer i, let M_i be $P^{-1}[N_i]$. Then for each positive integer i, M_i is a compact 3-manifold-with-boundary and $M_{i+1} \subset \text{Int } M_i$. Further, $\bigcap_{i=1}^{\infty} M_i = \text{Cl } H_G$.

Suppose that g is a nondegenerate element of G. For each positive integer i, let K_i be the component of M_i containing g. Then for each positive integer i, K_i is a compact 3-manifold-with-boundary, and $K_{i+1} \subset \text{Int } K_i$. Further $\bigcap_{i=1}^{\infty} K_i = g$.

Suppose now that j is a positive integer. We shall show that there is a 3-cell C such that $g \subset \text{Int } C$ and $C \subset \text{Int } K_j$. Let $\{R_1, R_2, \cdots, R_m\}$ be a finite set of 3-cells, each contained in Int K_j and such that $\{\text{Int } R_1, \text{Int } R_2, \cdots, \text{Int } R_m\}$ covers K_{j+1}. There exists a positive number ε such that if A is any subset of K_{j+1} of diameter less than ε, then A is contained in some one of Int R_1, Int R_2, $\cdots$, and Int R_m. By hypothesis, there is a homeomorphism f from E^3 onto E^3 such that (1) if $x \notin \text{Int } M_{j+1}$, then $f(x) = x$ and (2) $(\text{diam} f[g]) < \varepsilon$. It follows that $f[K_{j+1}] \subset K_{j+1}$, and hence there is a positive integer k not greater than m such that $f[g] \subset \text{Int } R_k$. Let C be $f^{-1}[R_k]$; clearly C is a 3-cell such that $g \subset \text{Int } C$ and $C \subset \text{Int } K_j$.

It now follows that g is cellular. There exists a 3-cell C_1 such that $g \subset \text{Int } C_1$ and $C_1 \subset \text{Int } K_1$. Let n_1 be 1. There is a positive integer n_2 such that $K_{n_2} \subset \text{Int } C_1$. There is a 3-cell C_2 such that $g \subset \text{Int} C_2$ and $C_2 \subset \text{Int} K_{n_2}$. A continuation of this process yields an increasing sequence $n_1, n_2, n_3, \cdots$ of positive integers and a sequence $C_1, C_2, C_3, \cdots$ of 3-cells such that for each positive integer i, $g \subset \text{Int } C_i$ and $C_i \subset \text{Int } K_{n_i}$. It is clear that for each positive integer i, $C_{i+1} \subset \text{Int } C_i$, and that $\bigcap_{i=1}^{\infty} C_i = g$. Hence g is cellular. Hence g is point-like and Theorem 5 is established.

The following theorem is closely related to Theorem 1. In the proof of Theorem 6 we construct triangulations in the order opposite to that used in the proof of Theorem 1.

THEOREM 6. *Suppose that G is a monotone decomposition of E^3 such that* (1) *E^3/G is homeomorphic to E^3 and* (2) Cl $P[H_G]$ *is a compact* 0-*dimensional set. Suppose that M is a compact polyhedral* 3-*manifold-with-boundary in E^3 such that* Bd M *is a connected and is disjoint from H_G. Suppose that M has a triangulation T such that the carrier Σ_1 of the* 1-*skeleton of T is disjoint from* Cl H_G. *Then there is a homeomorphism h from M onto $P[M]$ such that* $h \mid \text{Bd } M = P \mid \text{Bd } M$.

Proof. Let $\Delta_1, \Delta_2, \cdots$, and Δ_n be the 2-simplexes of T that intersect Int M. If $i = 1, 2, \cdots$, or n, let F_i be a disc in Int Δ_i and such that Int F_i contains $\Delta_i \cap \text{Cl} H_G$. If $i = 1, 2, \cdots$, or n, there is an annulus A_i on Δ_i such that

$$\text{Bd } A_i = (\text{Bd } \Delta_i) \cup (\text{Bd } F_i);$$

let γ_i be a centerline of A_i and let D_i be the subdisc of Δ_i bounded by γ_i. Note that $D_i \cap A_i$ is an annulus B_i, and that A_i is disjoint from Cl H_G.

Let N be $P[M]$; it may be shown that N is a compact 3-manifold-with-boundary, Bd $N = P[\text{Bd } M]$, and Int $N = P[\text{Int } M]$. If $i = 1, 2, \cdots$, or n, then $P[D_i]$ is a singular disc lying in Int N, and having no singularities on $P[B_i]$. Further, there is a neighborhood U of Σ_1 such that (1)$P \mid [U \cup (\bigcup_{i=1}^{n} A_i)]$ is a homeomorphism and (2) if $x \notin U \cup (\bigcup_{i=1}^{n} A_i$, then $P(x) \notin P[U \cup (\bigcup_{i=1} A_i)]$.

With the aid of Dehn's lemma [18], it may be shown that there exists discs $K_1, K_2, \cdots$, and K_n such that (1) if $i = 1, 2, \cdots$, or n, (a) $K_i \subset \text{Int } N$, (b) Bd $K_i = P[\text{Bd } D_i]$, and (c) $K_i \cap P[U \cup (\bigcup_{j=1}^{n} A_j)] \subset P[B_i]$, and (2) if i and j are distinct positive integers, neither greater than n, then K_i and Bd K_j are disjoint.

The discs $K_1, K_2, \cdots$, and K_n are not necessarily mutually disjoint. By an argument similar to that given in the proof of Theorem 9 of [7], it may be shown that there exist mutually disjoint discs $K_1', K_2', \cdots$, and K_n' such that if $i = 1, 2, \cdots$, or n, Bd K_i' = Bd K_i, $K_i' \subset$ Int N, and K_i' is disjoint from $P[\bigcup_{j=1}^{n}(A_j - B_j)]$. Then if $i = 1, 2, \cdots$, or n, $K_i' \cup P[A_i - B_i]$ is a disc Δ_i'.

By an argument similar to that used in the proof of Theorem 1, it may be shown that there is a homeomorphism h from M onto $P[M]$ such that $h \mid \text{Bd}\, M = P \mid \text{Bd}\, M$. This completes the proof of Theorem 6.

COROLLARY 2. *Suppose that G is a monotone decomposition of E^3 such that* (1) *E^3/G is homeomorphic to E^3 and* (2) *$\text{Cl}\, P[H_G]$ is a compact 0-dimensional set. Suppose that there exists a sequence $M_1, M_2, M_3, \cdots$ of compact 3-manifolds-with-boundary in E^3 such that* (1) *if n is any positive integer, $M_n \subset M_{n+1}$ and M_n has a triangulation such that the carrier of its 1-skeleton is disjoint from $\text{Cl}\, H_G$, and* (2) *$\bigcap_{i=1}^{\infty} M_i = \text{Cl}\, H_G$. Then each element of G is point-like.*

Proof. By an argument similar to that given to prove Theorem 2, but using Theorem 6 in place of Theorem 1, we may show that under the hypothesis of the corollary, the following holds: If U is any open set in E^3 containing H_G and ε is any positive number, there exists a homeomorphism h from E^3 onto E^3 such that (1) if $x \notin U$, $h(x) = x$, and (2) if g is any nondegenerate element of G, then $(\text{diam}\, h\,[g]) < \varepsilon$. It then follows by Theorem 5 that each element of G is point-like.

6. **Decompositions of E^3 into continua of type T.** The statement that a compact metric continuum M is of *type T* means that if K is any subcontinuum of M that can be embedded in a 2-sphere and f is an embedding of K in some 2-sphere S, then $f[K]$ does not separate S. Since for planar continua, separating the plane is a topological invariant, the criterion used is meaningful. Continua of type T include tree-like continua [8], snake-like continua [8], dendrons, and arcs.

THEOREM 7. *If G is a monotone decomposition of E^3 such that* (1) *E^3/G is homeomorphic to E^3,* (2) *$P[H_G]$ is contained in a compact 0-dimensional set, and* (3) *each element of G is a continuum of type T, then each element of G is point-like.*

Proof. First it will be shown that if M is a compact polyhedral 3-manifold-with-boundary in E^3 such that Bd M is connected and is disjoint from $\text{Cl}\, H_G$, then M has a triangulation L such that the carrier of the 1-skeleton of L is disjoint from $\text{Cl}\, H_G$.

Let L_0 be any triangulation of M. Let $v_1, v_2, \cdots$, and v_n denote the vertices of L_0. Since $\text{Cl}\, P[H_G]$ is compact and 0-dimensional, there are points $v_1', v_2', \cdots$, and v_n' of Int M not in $\text{Cl}\, H_G$. There is a homeomorphism h from M onto M such that $h \mid \text{Bd}\, M$ is the identity and if $i = 1, 2, \cdots$, or n, $h(v_i) = v_i'$. Let L_1 denote $\{h[\sigma]\colon \sigma \in L_0\}$. L_1 is a triangulation of M and no vertex of L_1 belongs to $\text{Cl}\, H_G$.

If s is a 1-simplex of L_1, let C_s be a 3-cell containing $s \cap \text{Cl}\, H_G$, obtained by a slight thickening of a subarc s' of s and such that $C_s \cap s = s'$. It is to be true that if s and t are distinct 1-simplexes of L_1, C_s and C_t are disjoint. Now by hypothesis, if g_0 is any subcontinuum of an element of G, g_0 does not separate any

2-sphere containing g_0. In addition, each component of $(\mathrm{Bd}\, C_s) \cap \mathrm{Cl}\, H_G$ is a subcontinuum of some element of G. Hence no component of $(\mathrm{Bd}\, C_s) \cap \mathrm{Cl}\, H_G$ separates $\mathrm{Bd}\, C_s$. By unicoherence, $(\mathrm{Bd}\, C_s) - (\mathrm{Cl}\, C_s)$ is connected. There is, therefore, an arc s'' on $\mathrm{Bd}\, C_s$, disjoint from $\mathrm{Cl}\, H_G$ and having as endpoints the endpoints of s'. It is now easy to construct a triangulation L of M such that the carrier of the 1-skeleton of L is disjoint from $\mathrm{Cl}\, H_G$.

By using Theorem 6 in place of Theorem 1, the argument used to establish Theorem 2 shows that the hypothesis of Theorem 5 is satisfied. Hence by Theorem 5, each element of G is point-like.

Corollary 3. *If G is a monotone decomposition of E^3 such that* (1) *E^3/G is homeomorphic to E^3,* (2) *$P[H_G]$ is contained in a compact 0-dimensional set, and* (3) *each element of G is a tree-like continuum, then each element of G is point-like.*

Corollary 4. *If G is a monotone decomposition of E^3 into arcs and one-point sets such that E^3/G is homeomorphic to E^3 and $\mathrm{Cl}\, P[H_G]$ is a compact 0-dimensional set, then each element of G is point-like* (2).

7. **Results on point-like decompositions.** In this section, we present some results on point-like decompositions of E^3. We begin with two results on 3-cells-with-handles.

The statement that K is a *3-cell-with-handles* means that K is an orientable 3-manifold-with-boundary such that there exist 3-cells $C_0, C_2, \cdots,$ and C_n such that (1) any two of $C_1, C_2 \cdots$ and C_n are disjoint, (2) if $i = 1, 2, \cdots,$ or n, $C_0 \cap C_i = (\mathrm{Bd}\, C_0) \cap (\mathrm{Bd}\, C_i)$ and $C_0 \cap C_i$ is the union of two disjoint discs, and (3) $K = \bigcup_{i=0}^{n} C_i$.

Lemma 3. *If K is a polyhedral 3-cell-with-handles and L is a polyhedral 3-cell such that $K \cap L = (\mathrm{Bd}\, K) \cap (\mathrm{Bd}\, L)$ and $K \cap L = D_1 \cup D_2 \cup \cdots \cup D_n$ where $D_1, D_2, \cdots,$ and D_n are mutually disjoint polyhedral discs, then $K \cup L$ is a polyhedral 3-cell-with-handles.*

Lemma 4. *If B is a compact 0-dimensional subset of E^3 and U is any open set containing B, there exists a polyhedral 3-manifold-with-boundary M such that $B \subset \mathrm{Int}\, M$, $M \subset U$, and each component of M is a 3-cell-with-handles.*

Proof. There exists a polyhedral 3-manifold-with-boundary N such that $B \subset N$ and $N \subset U$. It follows with the aid of Lemma 1 that there is a triangulation T of N such that the carrier of the 1-skeleton of T is disjoint from B.

Let N_0 be a component of N, and let S_0 be the carrier of the 1-skeleton of the triangulation T_0 of N_0 induced by T. There exists a polyhedral tubular neighborhood S_0^* of S_0 such that (1) S_0^* is disjoint from B, (2) if σ and σ' are distinct

(2) Joseph M. Martin has recently established this result without requiring that $\mathrm{Cl}\, P[H_G]$ be compact and 0-dimensional.

3-simplexes of T_0 and $\sigma - (\text{Int } S_0^*)$ and $\sigma' - (\text{Int } S_0^*)$ intersect, then their common part is a disc lying in the interior of some 2-simplex of T_0, and (3) $N_0 - \text{Int } S_0^*$ is connected. With the aid of Lemma 3, it follows that $N_0 - \text{Int } S_0^*$ is a 3-cell-with-handles. Lemma 4 now follows easily.

Suppose that G is an upper semicontinuous decomposition of E^3. The statement that H_G is *definable by* 3-*cell-with-handles* means that there exists a sequence $M_1, M_2, M_3, \cdots$ such that (1) for each positive integer n, M_n is a polyhedral 3-manifold-with-boundary such that each component of M_n is a 3-cell-with-handles and $M_{n+1} \subset \text{Int } M_n$, (2) $\text{Cl } H_G = \bigcap_{i=1}^{\infty} M_i$, and (3) g is a nondegenerate element of G if and only if g is a nondegenerate component of $\bigcap_{i=1}^{\infty} M_i$. It is clear that if H_G is definable by 3-cells-with-handles, then $\text{Cl } P[H_G]$ is a compact 0-dimensional set.

THEOREM 8. *Suppose that G is a point-like decomposition of E^3 such that* (1) *E^3/G is homeomorphic to E^3 and* (2) $\text{Cl } P[H_G]$ *is a compact* 0-*dimensional set. Then H_G is definable by* 3-*cells-with-handles.*

Proof. Since E^3/G is homeomorphic to E^3, there is, by Lemma 4, a sequence $M_1, M_2, M_3, \cdots$ such that (1) for each positive integer i, M_i is a compact polyhedral 3-manifold-with-boundary, $M_{i+1} \subset \text{Int } M_i$, and each component of M_i is a 3-cell-with-handles, and (2) $\text{Cl } P[H_G] = \bigcap_{i=1}^{\infty} M_i$. It is clear that $\text{Cl } H_G = \bigcap_{i=1}^{\infty} P^{-1}[M_i]$.

Suppose that j is a positive integer and L is a component of M_j. Since L is a 3-cell-with-handles, there exist polyhedral 3-cells $C_0, C_1, \cdots$, and C_r such that (1) any two of $C_1, C_2, \cdots$, and C_r are disjoint, (2) if $i = 1, 2, \cdots$, or r,

$$C_0 \cap C_i = (\text{Bd } C_0) \cap (\text{Bd } C_i)$$

and $C_0 \cap C_1$ is the union of two disjoint polyhedral discs D_i' and D_i, and (3) $L = \bigcup_{i=0}^{r} C_i$. An argument similar to that given in proving Theorem 1 may be used to show that there exists 3-cells $C_0', C_1', \cdots$, and C_r' such that $\bigcup_{i=0}^{r} C_i' = P^{-1}[L]$ and $C_0', C_1', \cdots$, and C_r' satisfy conditions relative to $P^{-1}[L]$ analogous to those satisfied by $C_0, C_1, \cdots$, and C_r relative to L. It follows that $P^{-1}[L]$ is a 3-cell-with-handles. It is easily seen that H_G is definable by 3-cells-with-handles.

The following result, which was announced in [3], provides a converse to Theorem 8. It may be proved by line of argument similar to that used to establish Theorem 6 and Corollary 2.

THEOREM 9. *Suppose that G is a monotone decomposition of E^3 such that* (1) *E^3/G is homeomorphic to E^3 and* (2) $P[H_G]$ *is a compact* 0-*dimensional set. If H_G is definable by* 3-*cells-with-handles, then each element of G is point-like.*

Our last result concerns the construction of a point-like decomposition of E^3 for which the associated decomposition space has certain given properties.

THEOREM 10. *If K is any compact 0-dimensional subset of E^3, there exist a point-like decomposition G of E^3 and a homeomorphism h from E^3/G onto E^3 such that* $h[P[H_G]] = K$.

Proof. Let $M_1, M_2, M_3, \cdots$ be a sequence of polyhedral 3-manifolds-with-boundary such that (1) if i is any positive integer, $M_{i+1} \subset \text{Int } M_i$, and (2) $\bigcap_{i=1}^{\infty} M_i = K$. Let P_1 and P_2 be disjoint planes in E^3.

There is a homeomorphism h_1 from E^3 onto E^3 such that if M_1' is any component of M_1, $h_1[\text{Int } M_1']$ intersects both P_1 and P_2. There is a homeomorphism h_2 from E^3 onto E^3 such that if $x \in E^3 - \text{Int } M_1$, $h_2(x) = x$ and if M_2' is any component of M_2, then $h_2 h_1[\text{Int } M_2']$ intersects both P_1 and P_2. Suppose that i is a positive integer and $h_1, h_2, \cdots$, and h_i have been defined. There is a homeomorphism h_{i+1} from E^3 onto E^3 such that if $x \in E^3 - \text{Int } M_i$, $h_{i+1}(x) = x$, and if M_{i+1}' is any component of M_{i+1}, then $h_{i+1} h_i \cdots h_1[\text{Int } M_{i+1}']$ intersects both P_1 and P_2. Therefore there exists a sequence $h_1, h_2, h_3, \cdots$ of homeomorphisms from E^3 onto E^3 having properties indicated above. For each positive integer i, let N_i be $h_i h_{i-1} \cdots h_1[M_i]$.

Let G be the decomposition of E^3 whose nondegenerate elements are the components of $\bigcap_{i=1}^{\infty} N_i$. It may be proved, using [17, Chapter V, Theorem 20], that G is upper semicontinuous. Note that $H_G = \bigcap_{i=1}^{\infty} N_i$.

Now we shall show that if U is any open set in E^3 containing H_G and ε is any positive number, there is a homeomorphism f from E^3 onto E^3 such that if $x \notin U$, $f(x) = x$ and if g is any nondegenerate element of G, (diam $f[g]) < \varepsilon$. Suppose U is open in E^3 and ε is any positive number. There is a positive integer j such that $N_j \subset U$. Since (1) for each positive integer n, M_n is compact, (2) $h_j h_{j-1} \cdots h_1$ is a homeomorphism, and (3) the maximum of the diameters of the components of M_n approaches 0 as n increases without bound, there is a positive integer k greater than j such that if M_k' is any component of M_k', (diam $h_j h_{j-1} \cdots h_1 [M_k']) < \varepsilon$. Let f be $(h_k h_{k-1} \cdots h_{j+1})^{-1}$. Then $f[N_k] = h_j h_{j-1} \cdots h_1[M_k]$ and if $x \in E^3 - N_j$, then $f(x) = x$. Hence f is a homeomorphism from E^3 onto E^3 having the specified properties.

It may now be seen, from a consideration of the proof of Theorem 1 of [4], that there is a homeomorphism h from E^3/G onto E^3 such that $h[P[H_G]] = K$. With the aid of Theorem 5, it may be proved that each element of G is point-like.

REFERENCES

1. S. Armentrout, *Upper semi-continuous decompositions of E^3 with at most countably many non-degenerate elements*, Ann. of Math. **78** (1963), 605–618.

2. ———, *Concerning point-like decompositions of S^3 that yield 3-manifolds*, Abstract 619–115, Notices Amer. Math. Soc. **12** (1965), 90.

3. R. J. Bean, *Decompositions of E^3 which yield E^3*, Abstract 619–198, Notices Amer. Math. Soc. **12** (1965), 117.

4. R. H. Bing, *Upper semicontinuous decompositions of E^3*, Ann. of Math. **65** (1957), 363–374.

5. ———, *A decomposition of E^3 into points and tame arcs such that the decomposition space is topologically different from E^3*, Ann. of Math. **65** (1957), 484–500.

6. ————, *A homeomorphism between the 3-sphere and the sum of two solid horned spheres*, Ann. of Math. **59** (1952), 354–362.

7. ———, *Point like decompositions of E^2*, Fund. Math. **50** (1962), 431–453.

8. ———, *Snake-like continua*, Duke Math. J. **18** (1951), 653–663.

9. ———, *Inequivalent families of periodic homeomorphisms of E^3*, Ann. of Math. **80** (1964), 78–93.

10. ———, *Topology of 3-manifolds and related topics*, Decompositions of E^3, Prentice-Hall, Englewood Cliffs, N. J., 1962; pp. 5–21.

11. ———, *An alternative proof that 3-manifolds can be triangulated*, Ann. of Math. **69** (1959), 37–65.

12. K. W. Kwun, *Upper semi-continuous decompositions of the n-sphere*, Proc. Amer. Math. Soc. **13** (1962), 284–290.

13. K. W. Kwun and F. Raymond, *Almost acyclic maps of manifolds*, Amer. J. Math. **86** (1964), 638–650.

14. L. L. Lininger, *The sum of two crumpled cubes is S^3 if it is a 3-manifold*, Abstract 64T-445, Notices Amer. Math. Soc. **11** (1964), 678.

15. L. F. McAuley, *Another decomposition of E^3 into points and intervals* (to appear).

16. E. E. Moise, *Affine structures in 3-manifolds*. V. *The triangulation theorem and Hauptvermutung*, Ann. of Math. **56** (1952), 92–114.

17. R. L. Moore, *Foundations of point set theory*, rev. ed., Amer. Math. Soc. Colloq. Publ. Vol. 13, Amer. Math. Soc., Providence, R. I., 1962,

18. C. D. Papakyriakopoulos, *On Dehn's lemma and the asphericity of knots*, Ann. of Math. **66** (1957), 1–250.

19. T. M. Price, *Cellular decompositions of E^3*, Ph. D. Thesis, University of Wisconsin, Madison, Wis., 1964.

20. H. Schubert, *Knoten und Vollringe*, Acta Math. **90** (1953), 132–286.

21. D. G. Stewart, *Cellular subsets of the 3-sphere*, Trans. Amer. Math. Soc. **114** (1965), 10–22.

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